

Appendix

In this appendix we prove that, in the case of the lognormal distribution, $\lim_{y \rightarrow \infty} \eta_\theta = -\infty$.¹² Writing out the whole expression, we get

$$\begin{aligned} \eta_\theta(y, \theta) = & \frac{1}{8} \sqrt{2} \frac{\exp\left(-\frac{1}{8} \frac{(2 \ln y + \ln \theta)^2}{\ln \theta}\right)}{\ln^3 \theta} \left\{ \frac{4 \left(\ln^{\frac{1}{2}} \theta\right) \pi \ln^2 y - 4 \left(\ln^{\frac{1}{2}} \theta\right) \pi \left(\ln^2 y\right) \Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right)}{\theta \pi^{3/2} \left(\Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right) - 1\right)^2} + \right. \\ & - \frac{\left(\ln^{\frac{5}{2}} \theta\right) \pi - \left(\ln^{\frac{5}{2}} \theta\right) \pi \Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right)}{\theta \pi^{3/2} \left(\Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right) - 1\right)^2} - \frac{4 \left(\ln^{\frac{3}{2}} \theta\right) \pi - 4 \left(\ln^{\frac{3}{2}} \theta\right) \pi \Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right)}{\theta \pi^{3/2} \left(\Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right) - 1\right)^2} \\ & \left. + \frac{2 \sqrt{2} \exp\left(-\frac{1}{8} \frac{(2 \ln y + \ln \theta)^2}{\ln \theta}\right) (\ln^2 \theta) \sqrt{\pi} - 4 \sqrt{2} \exp\left(-\frac{1}{8} \frac{(2 \ln y + \ln \theta)^2}{\ln \theta}\right) (\ln \theta) \sqrt{\pi} \ln y}{\theta \pi^{3/2} \left(\Phi\left(\frac{1}{4} \sqrt{2} \frac{2 \ln y + \ln \theta}{\ln\left(\frac{1}{2}\right) \theta}\right) - 1\right)^2} \right\} \end{aligned}$$

This function can be written as

$$\frac{\sqrt{2}}{8} \frac{e^{-z^2}}{\alpha^6} \left[\frac{4 \alpha \pi (1 - \Phi(z)) \ln^2 y - \alpha^5 \pi (1 - \Phi(z)) - 4 \alpha^3 \pi (1 - \Phi(z)) + 2 \alpha^2 \sqrt{2 \pi} e^{-z^2} (\alpha^2 - 2 \ln y)}{\theta \pi^{\frac{3}{2}} (\Phi(z) - 1)^2} \right]$$

where $z = \frac{\sqrt{2}}{4} \frac{2 \ln y + \ln \theta}{\alpha}$ and $\alpha = \sqrt{\ln \theta}$, so that $\ln y = \sqrt{2} \alpha z - \frac{\alpha^2}{2}$. Substituting for the latter, we get

$$\frac{\sqrt{2}}{8} \frac{e^{-z^2}}{\alpha^6} \left[\frac{\alpha \pi (1 - \Phi(z)) (8 \alpha^2 z^2 - 4 \sqrt{2} \alpha^3 z - 4 \alpha^2) + 4 \sqrt{2} \alpha^3 e^{-z^2} (\alpha - \sqrt{2} z)}{\theta \pi^{\frac{3}{2}} (\Phi(z) - 1)^2} \right]$$

which is in terms of z , and collapses to

$$\begin{aligned} & \frac{\sqrt{2}}{2} \frac{e^{-z^2}}{\alpha^3} \left[\frac{\pi (1 - \Phi(z)) (2 z^2 - \sqrt{2} \alpha z - 1) - \sqrt{2 \pi} e^{-z^2} (\sqrt{2} z - \alpha)}{\theta \pi^{\frac{3}{2}} (\Phi(z) - 1)^2} \right] \\ & = \frac{e^{-z^2} \sqrt{2 \pi}}{2 \theta \alpha^3 \pi^{\frac{3}{2}}} \left[\frac{\sqrt{\pi} (1 - \Phi(z)) (2 z^2 - \sqrt{2} \alpha z - 1) - e^{-z^2} (2 z - \sqrt{2} \alpha)}{(1 - \Phi(z))^2} \right] \quad (\text{a.1}) \end{aligned}$$

¹²We owe very useful suggestions about this proof to profs A.Leaci and D.Scolozzi.

To ease notation, let $k = \frac{\sqrt{2\pi}}{2\theta\alpha^3\pi^{\frac{3}{2}}}$, a constant. Then (a.1) can be written as

$$\begin{aligned}
& ke^{-z^2} \left[\frac{(2z - \sqrt{2}\alpha) \left(\sqrt{\pi} (1 - \Phi(z)) z - e^{-z^2} \right) - \sqrt{\pi} (1 - \Phi(z))}{[1 - \Phi(z)]^2} \right] \\
&= \frac{ke^{-z^2}}{1 - \Phi(z)} \left\{ \frac{2z [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))} - \frac{\sqrt{2}\alpha [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))} - \sqrt{\pi} \right\} \quad (\text{a.2})
\end{aligned}$$

Thus finding the $\lim_{y \rightarrow \infty} \eta_\theta = -\infty$ amounts to studying the behaviour of (a.2) as z tends to infinity. We now consider the curly brackets of (a.2), to calculate

$$\lim_{z \rightarrow \infty} \frac{2z [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))} \quad \text{and} \quad \lim_{z \rightarrow \infty} \frac{\sqrt{2}\alpha [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))}$$

As to the former, we can use Hôpital's rule to obtain

$$\lim_{z \rightarrow \infty} \frac{2z [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))} = 2 \lim_{z \rightarrow \infty} \frac{\frac{2\sqrt{\pi}(1 - \Phi(z))z}{e^{-z^2}} - 1}{-\frac{2}{\sqrt{\pi}}} = -\sqrt{\pi} \quad (\text{a.3})$$

where we use the fact that, by Hôpital's rule,

$$\lim_{z \rightarrow +\infty} \frac{(1 - \Phi(z))z}{e^{-z^2}} = \frac{1}{\sqrt{\pi}} \quad (\text{a.4})$$

As to the latter, notice that

$$\lim_{z \rightarrow \infty} \frac{\sqrt{2}\alpha [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))} = \lim_{z \rightarrow \infty} \left\{ \frac{\sqrt{2}\alpha}{z} \frac{z [\sqrt{\pi}(1 - \Phi(z))z - e^{-z^2}]}{(1 - \Phi(z))} \right\} = 0 \quad (\text{a.5})$$

where we use (a.3).

Finally, the term outside the curly brackets tends to infinity. To see this, notice that

$$\lim_{z \rightarrow \infty} \frac{ke^{-z^2}}{1 - \Phi(z)} = k \lim_{z \rightarrow \infty} \frac{e^{-z^2}}{z(1 - \Phi(z))} \cdot z = \infty$$

using (a.4). Putting together these limits, $\lim_{y \rightarrow \infty} \eta_\theta = -\infty$.

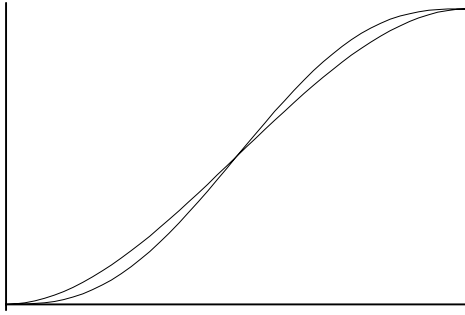


Fig. 1a. Single crossing of distributions

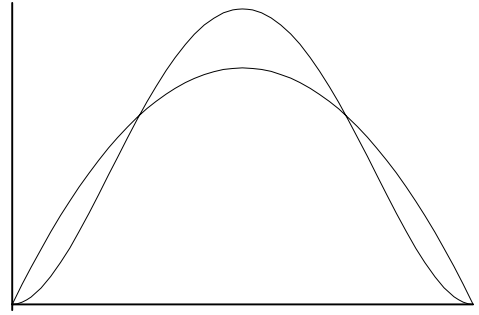


Fig. 1b. Double-crossing of densities

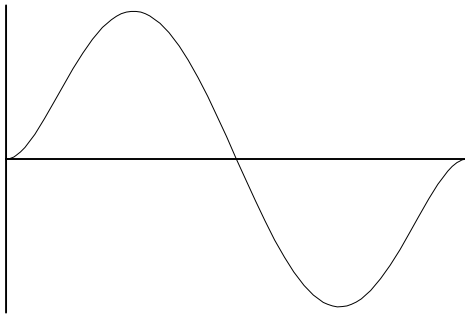


Fig. 1c. The function $F_{\theta}(y, \theta)$

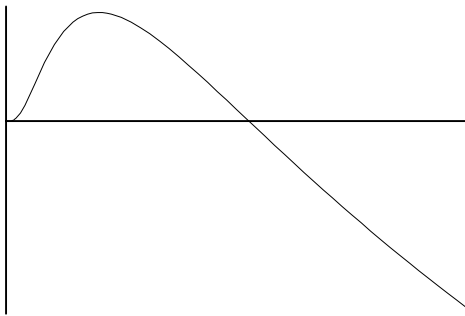


Fig. 2. The function $\eta_{\theta}(p, \theta)$